DIRICHLET PROBLEMS OF A QUASI-LINEAR ELLIPTIC SYSTEM

GONGBO LI AND LI MA

ABSTRACT. We discuss the Dirichlet problem of the quasi-linear elliptic system

$$\begin{aligned} -e^{-f(U)} div(e^{f(U)} \bigtriangledown U) + \quad \tfrac{1}{2} f'(U)|\bigtriangledown U|^2 &= 0, \quad \text{in } \Omega, \\ U|_{\partial\Omega} &= \phi. \end{aligned}$$

Here Ω a smooth bounded domain in R^n, $f : R^N \to R$ is a smooth function, $U : \Omega \to R^N$ is the unknown vector-valued function, $\phi : \overline{\Omega} \to R^N$ is a given vector-valued C^2 function, f' is the gradient of the function f with respect to the variable U. Such problems arise in population dynamics and Differential Geometry. The difficulty of studying this problem is that this nonlinear elliptic system does not fit the usual growth condition in M.Giaquinta's book [G] and the natural working space $H^1 \cap L^\infty(\Omega)$ for the corresponding Euler-Lagrange functional does not fit the usual minimization or variational argument. We use the direct method on a convex subset of $H^1 \cap L^\infty(\Omega)$ to overcome these difficulties. Under a suitable assumption on the function f, we prove that there is at least one solution to this problem. We also give application of our result to the Dirichlet problem of harmonic maps into the standard sphere

1. Introduction

In the interesting paper, Bensoussan-Boccardo-Murat [BBM] studied a class of quasi-linear elliptic equations which include the following equation as special case:

$$-\Delta u + u|Du|^2 = h(x), \text{ in } \Omega \quad u|_{\partial\Omega} = 0. \tag{$*$}$$

where Ω a smooth bounded domain in R^n with $h \in L^\infty()\Omega$. Observe that equation (*) has a variational structure for left side of (*). In fact, let $\rho(u) = -u^2/2$. Then the equation can be written as

$$-e^{-\rho(u)} div(e^{\rho(u)} \bigtriangledown u) = h(x), \text{ in } \Omega, \quad u|_{\partial\Omega} = 0.$$

Let $P(u) = \int_0^u e^{\rho(s)} ds$. Clearly $P = P(u)$ has its derivative $P'(u) = e^{\rho(s)} > 0$, and has its inverse function $u = G(v)$. Define $v = P(u)$. Then the equation can be reduced into

$$-\Delta v = e^{-G(v)^2/2} h := h(x, v), \text{ in } \Omega, \quad v|_{\partial\Omega} = 1.$$

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In this case, the nonlinear term is in right side and it is very tamed. The variational structure is

$$J(v) = \frac{1}{2}\int_\Omega |Dv|^2 - \int_\Omega H(x,v)$$

where $H(x,v) = \int^v h(x,s)ds$. Using the mountain pass method or variational techniques one can easily prove the existence of a classical positive solution of the problem (*). This kind of idea is usually called the Cole-Hopf transformation in the study of parabolic equations. One may also obtain a Liouville type theorem for bounded smooth solutions of the problem:

$$-e^{-\rho(u)} div(e^{\rho(u)} \bigtriangledown u) = 0, \text{ in } R^n.$$

If we consider the evolution equation of the following form:

$$u_t - \Delta u + u|Du|^2 = h(x), \text{ in } \Omega, \quad u|_{t=0} = \varphi, \quad u|_{\partial\Omega} = 0. \tag{**}$$

We can also use the trick above and reduced this equation into

$$v_t - \Delta v = e^{-G(v)^2/2}h, \text{ in } \Omega, \quad v|_{t=0} = P(\varphi), \quad v|_{\partial\Omega} = 1.$$

Hence one can also get a global existence of solutions of the problem (**). However, this trick is not so good for quasi-linear elliptic systems or parabolic systems. We can also consider (*) in another way. Let $m(u) = -u^2$. Then the elliptic equation in (*) can be reduced into

$$-e^{-m(u)} div(e^{m(u)} \bigtriangledown u) + \frac{1}{2}m'(u)|\bigtriangledown u|^2 = h(x)$$

and the left side is the variational derivative of the functional

$$j(u) = \int_\Omega e^{m(u)}|Du|^2 dx.$$

This tells us that we can use this functional to study the corresponding Dirichlet boundary value problem:

$$-e^{-m(u)} div(e^{m(u)} \bigtriangledown u) + \frac{1}{2}m'(u)|\bigtriangledown u|^2 = 0, \text{ in } \Omega \quad u|_{\partial\Omega} = \varphi.$$

Although the functional $j(\cdot)$ is not so nice, one can show that this problem is always solvable.

In this paper, we mainly discuss the Dirichlet problem of the quasi-linear elliptic system

$$-e^{-f(U)} div(e^{f(U)} \bigtriangledown U) + \frac{1}{2}f'(U)|\bigtriangledown U|^2 = 0, \text{ in } \Omega, \quad U|_{\partial\Omega} = \phi. \tag{1.1}$$

Here again Ω a smooth bounded domain in R^n, $f : R^N \to R$ is a smooth function, $U : \Omega \to R^N$ is the unknown vector-valued function, $\phi : \partial\Omega \to R^N$ is a given

vector-valued $C^{2,\delta}$ function, f' is the gradient of the function f with respect to the variable U. Such problems arise in population dynamics and Differential Geometry (see below). The difficulty of this problem is that this nonlinear elliptic system does not fit the usual growth condition in M.Giaquinta's book [G] (see also [C] and [S]) and the natural working space $H^1 \cap L^\infty(\Omega)$ for the corresponding Euler-Lagrange functional does not fit the variational argument. We point out that the space $H^1 \cap L^\infty(\Omega)$ or $W^{1,q}$ $(q > 2)$ is the right for the corresponding variational functional to be differentiable. Once we have this, we can derive the Euler-Lagrange system for the functional. In our earlier research [M], we noticed that there is one nice direction for us to differentiate the variational functional. Then we use this property and the Nash-De Giorgi- Moser-Morrey iteration to obtain such a L^∞ bound. This method is tricky, we used it in the study of certain elliptic systems (see [L]). However, it is not so powerful. In fact, we can not use it to treat the system (1.1). Our idea here is to find a nice space obtaining a L^∞ bound. To get such a L^∞ bound, we use the direct method on a convex subset of $H^1 \cap L^\infty(\Omega)$. In the following we will study the existence of a weak solution of (1.1) in the class $H^1 \cap L^\infty(\Omega)$.

Theorem 1.1. *Suppose that $f(U)$ satisfies that $f'(U) = -Ug(U)$, where $g(U)$ is a positive continuous function on R^N. Then there is at least one weak bounded solution U of (1.1). Furthermore, there is some $q > 2$ such that $U \in W^{1,q}(\Omega)$.*

Typical examples satisfying our assumption of Theorem 1 are (1) $f(U) = -\alpha|U|^2$,(2) $f(U) = -2\log\frac{1+|U|^2}{2}$, and (3) $f(U) = -\beta\log\frac{1+|U|^2}{2}$ where $\alpha, \beta > 0$. Case (1) corresponds to Gaussian measure $dm = e^{-|Y|^2}dY$. In this case, $f'(Y) = -2\alpha Y$. Case (2) is for the standard metric $ds^2 = \frac{4}{(1+|Y|^2)^2}dY^2$on S^{N-1} written in the stereographic projection coordinates (Y) from the north pole. For this case, we have that $f'(Y) = -4Y/(1 + |Y|^2)$. So the critical map of $E(\cdot)$ is in fact a harmonic map from Ω to the sphere. Then the result was obtained by S.Hildebrandt, H.Kaul, and K.O.Widman [HKW], Schoen and Uhlenbeck in [SU] and Giaquinta and Giusti in [GG]. When our f satisfies some convexity condition, one may consult the papers of S. Hildebrandt [H], D.G.Defigueiredo [D], and Marcellini-Sbordone, [MS]. As a by-product, we can obtain the following extension of a result due to V.Benci and J.M.Coron in dimension two [BC].

Corollary 1.2. . *Given a bounded domain Ω in R^n with regular (Lipschitz) boundary. Assume we have a $C^{2,\delta}$ map $\phi : \partial\Omega \to S^N$. Auume $n \leq N$. Then these exist at least two weakly harmonic maps from $\Omega \to S^N$ with Dirichlet boundary value ϕ.*

Proof. Since $n \leq N$ and $\phi : \partial\Omega \to S^N$ is smooth, we have find a point P in S^N such that $\{P, -P\}$ are not in the range $\phi(\partial\Omega)$. Taking P as the north pole and using the

stereographic coordinates at P, we can reduce the harmonic map problem into our problem (1.1) with $f(U) = -2\log\frac{1+|U|^2}{2}$. By Theorem 1.1, we have a weak bounded solution U, which corresponds a weakly harmonic map from $\Omega \to S^N$ with Dirichlet boundary value ϕ. In the same way, using $-P$ as south pole, we can obtain another weakly harmonic map from $\Omega \to S^N$ with Dirichlet boundary value ϕ. $\square$

Our method can be used to handle the following Dirichlet problem of the quasi-linear elliptic system

$$-e^{-f(U)}\partial_i(e^{f(U)}A(x)^{ab}_{ij}\partial_j U^a) + \frac{1}{2}[f'(U)]^b A(x)^{ac}_{ij}\partial_i U^a \partial_j U^c = 0, \text{ in } \Omega, \quad U|_{\partial\Omega} = \phi. \tag{1.1'}$$

Here we use the Einstein sum convention and we assume that (A^{ab}_{ij}) is a uniformly positive matrix function in the sense that its each component is non-negative and there are two positive constants λ and Λ such that

$$\lambda|\xi|^2 \leq A(x)^{ab}_{ij}\xi^i_a\xi^j_b \leq \Lambda|\xi|^2, \tag{1.a}$$

for any $\xi \in R^n \times R^N$. The energy integral for this problem is

$$E(U) = \int_\Omega e^{f(U)}A^{ab}_{ij}\partial_i U^a \partial_j U^b dx$$

Our result for this problem is

Theorem 1.3. *Suppose that $f(U)$ satisfies that $f'(U) = -Ug(U)$, where $g(U)$ is a positive continuous function on R^N. Assume that the matrix function $(A^{ab}_{ij}(x))$ satisfies* (1.a). *Then there is at least one weak bounded solution U of (1.1'). Furthermore, there is some $q > 2$ such that $U \in W^{1,q}(\Omega)$.*

We also remark that we can extend our result to the Dirichlet boundary problem on half space R^n_+

$$-e^{-f(U)}div(e^{f(U)} \bigtriangledown U) + \frac{1}{2}f'(U)|\bigtriangledown U|^2 = 0, \text{ in } R^n_+, \quad U|_{x_n=0} = \phi. \tag{1.1''}$$

where ϕ is a smooth bounded function on $\overline{R^n_+}$.

We have the following result:

Theorem 1.4. *Suppose that $f(U)$ satisfies that $f'(U) = -Ug(U)$, where $g(U)$ is a positive continuous function on R^N. Assume that ϕ is a smooth bounded function on $\overline{R^n_+}$ with $\int_{R^n_+}|D\phi|^2dx < +\infty$. Then there is at least one weak bounded solution U of (1.1"). Furthermore, there is some $q > 2$ such that $U \in W^{1,q}_{loc}(R^n_+)$.*

Clearly one can extend our results to the case when $f = f(U)$ is replaced by a more general function $f = f(x, U)$. One may also discuss the Dirichlet problem

of a quasi-linear elliptic system with a p-Laplacian type operator on the smooth bounded domain Ω in R^n or in the half space R^n_+:

$$-L_pU = 0 \text{ in } \Omega, \quad U|_{\partial\Omega} = \phi. \tag{1.2}$$

Here, $p > 1$, and

$$L_pu := e^{-f(U)}div(e^{f(U)}|DU|^{p-1}DU) + \frac{1}{2}f'(U)|\bigtriangledown U|^2$$

with $f \in C^2(R^N)$. Since the formulations are complicated, we omit it in this paper.

2. Proofs of Theorem 1.1 and 1.3

We first prove Theorem 1.1:

We define the following integral

$$E(U) = \int_\Omega e^{f(U)}|DU|^2dx$$

Then one can formally compute that

$$< DE(U), V >= \int_\Omega e^{f(U)}(DU, DV) + \frac{1}{2}\int_\Omega e^{f(U)}f'(U)|DU|^2dx$$

Hence the Euler-Lagrange equation for E is (1.1).

Since ϕ has a smooth extension over $\overline{\Omega}$, we can find a constant vector C such that $-C \leq \phi(x) \leq C$ for each $x \in \overline{\Omega}$. Introduce the convex subset

$$\mathcal{A} = \{U \in H^1(\Omega); -C \leq U(x) \leq C, \text{in } \Omega, U(x) = \phi(x), \text{in } \partial\Omega\}$$

Clearly $\phi \in \mathcal{A}$ and $\mathcal{A}$ is a weakly closed convex subset of H^1. The functional E is weakly lower semi-continuous on $\mathcal{A}$ and C^1 on $H^1 \cap L^\infty(\Omega)$. Hence, we have a minimizer $U \in \mathcal{A}$ such that $E(U) = \inf_{v\in\mathcal{A}} E(v)$. We will prove that this U is a weak solution of (1.1). Then we prove Theorem 1.1.

Take any vector-valued function $\xi \in C^2_0(\Omega)$ and a small positive constant ϵ. Define

$$V_\epsilon(x) = min\{C, max\{-C, U(x) + \epsilon\xi(x)\}\}.$$

Here the maximum and minimum are taken for each component. Then we have

$$V_\epsilon \in \mathcal{A}.$$

Define

$$\Omega^\epsilon := \{x \in \Omega; U(x) + \epsilon\xi(x) \geq C > U(x)\}$$

and

$$\Omega_\epsilon := \{x \in \Omega; U(x) + \epsilon\xi(x) \leq -C < U(x)\}$$

Observe that for $\epsilon > 0$ small we have $U > 0$ on Ω^ϵ and $U < 0$ on Ω_ϵ . We also have the measure $|\Omega^\epsilon| \to 0$ and the measure $\Omega_\epsilon \to 0$ as $\epsilon \to 0+$.

Let

$$\xi^\epsilon(x) = max\{0, -C + U(x) + \epsilon\xi(x)\}.$$

and

$$\xi_\epsilon(x) = -min\{0, C + U(x) + \epsilon\xi(x)\}.$$

Clearly, $\xi^\epsilon(x) \geq 0$ and $\xi_\epsilon(x) \geq 0$, and both are in $H_0^1 \cap L^\infty(\Omega)$. Then we can rewrite V_ϵ as

$$V_\epsilon = U + \epsilon\xi - \xi^\epsilon + \xi_\epsilon.$$

Since, for $t \in [0, 1)$,

$$E(U + t(V_\epsilon - U)) \geq E(U),$$

we have

$$< DE(U), V_\epsilon - U >\geq 0.$$

Note that

$$< DE(U), V_\epsilon - U >=< DE(U), \epsilon\xi - \xi^\epsilon + \xi_\epsilon > .$$

Hence we have that

$$< DE(U), \xi >\geq \frac{1}{\epsilon}[< DE(U), \xi^\epsilon > - < DE(U), \xi_\epsilon >]$$

We will show that $< DE(U), \xi^\epsilon >\geq \circ(\epsilon)$ and $< DE(U), \xi_\epsilon >\geq \circ(\epsilon)$. In fact, we have,

$$\begin{aligned}
< DE(U), \xi^\epsilon > &= \int_\Omega e^{f(U)}(DU, D\xi)dx \\
&+ \frac{1}{2}\int_\Omega (e^{f(U)} f'(U) \cdot \xi^\epsilon |DU|^2 dx \\
&= \int_{\Omega^\epsilon} e^{f(U)}|DU|^2 + \frac{1}{2}\int_{\Omega^\epsilon} e^{f(U)} f'(U) \cdot (U - C)|DU|^2 \\
&+ \epsilon\int_{\Omega^\epsilon} e^{f(U)}(DU, D\xi) + \frac{1}{2}\epsilon\int_{\Omega^\epsilon} e^{f(U)} f'(U) \cdot \xi|DU|^2 \\
&= \int_{\Omega^\epsilon} e^{f(U)}|DU|^2 g(U)(1 - \frac{1}{2}U \cdot (U - C)) \\
&+ \epsilon\int_{\Omega^\epsilon} e^{f(U)}(DU, D\xi) + \frac{1}{2}\epsilon\int_{\Omega^\epsilon} e^{f(U)} f'(U) \cdot \xi|DU|^2 \\
&\geq \epsilon\int_{\Omega^\epsilon} e^{f(U)}(DU, D\xi) + \frac{1}{2}\epsilon\int_{\Omega^\epsilon} e^{f(U)} f'(U) \cdot \xi|DU|^2 = \circ(\epsilon).
\end{aligned}$$

In the last inequality we used the fact that for $\epsilon > 0$ small we have $U \cdot (C - U) > 0$ on Ω^ϵ (since $U > 0$ and $C - U > 0$ on Ω^ϵ), and in the last equality we used $|\Omega^\epsilon| \to 0$. Similarly we can prove that $< DE(U), \xi_\epsilon > \geq \circ(\epsilon)$.Therefore, we have

$$< DE(U), \xi > \geq \circ(1) \to 0+$$

Since our ξ is arbitrary, we have

$$< DE(U), \xi >= \circ(1) \to 0+$$

and

$$< DE(U), \xi >= 0.$$

Before finishing this section, we point out that our U is a spherical $Q-$ minima in the sense M.Giaquinta and Giusti [GG]. Hence we have $U \in W^{1,q}(\Omega)$ for some $q > 2$. This proves our Theorem 1.1.

The proof of Theorem 1.2 is a easy adaptation of the argument above. So we omit the detail.

3. Proof of Theorem 1.4

In this section we will give the proof of Theorem 1.3.

We choose a bounded domain exhaustion of R^n_+ in the way that $R^n_+ = U_{R>0}(R^n_+) \cap B_R(0)$. Here $B_R(0)$ is the ball of radius R with center at 0. We write $B^+_R = (R^n_+) \cap B_R(0)$.

For any bounded domain Ω in R^n_+, we let

$$E_\Omega(U) = \int_\Omega e^{f(U)} |DU|^2 dx$$

Fix a large $R > 0$. We define the following integral on B^+_R

$$E_R(U) = \int_{B^+_R} e^{f(U)} |DU|^2 dx$$

By Theorem 1.1, we have a bounded weak solution u_R with uniform bounds:

$$|u_R|_{L^\infty(B^+_R)} \leq |u_R|_{L^\infty(R^n_+)}$$

and

$$|Du_R|_{L^2(B^+_R)} \leq C_2 |D\phi|_{L^2(R^n_+))}$$

where C_2 is a unform constant.

With these two bounds we can use extracting a diagonal subsequence method to get a weakly convergence sequence (U_k) and a weak solution $W \in H^1_{loc}(R^n_+) \cap L^\infty(R^n_+)$ of (1.1")with the following property

$$E_\Omega(W) \leq \underline{lim}_{k\to\infty} E_\Omega(U_k).$$

This proves Theorem 1.3.

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Institute of Physics and Mathematical Sciences, WuHan, P.R. China
E-mail address: ligb@wipm.ac.cn

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, P.R. China
E-mail address: lma@math.tsinghua.edu.cn